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Improved Solution for Ill-Conditioned Algebraic Equations by Epsilon Decomposition

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Nomenclature

(A)	$= n \times n$ symmetric matrix
$(\bar{A})$	= matrix (A) with ϵ added to diagonal elements
$\{b\}$	= right-hand side of simultaneous algebraic equations
$\{c\},\{\bar{c}\}$	= coefficient vectors for eigenvector expansion of $\{x\}$ and $\{\bar{x}\}$
c_i	= elements of $\{c\}$
$K(A)$	= spectral condition number of (A)
p	= number of zero and near-zero eigenvalues
r	= rank of (A)
$\{x\}$	= solution vector
$\{\bar{x}\}$	= E-D solution before improvement
$\{x_{ED}\}$	= E-D solution after improvement
$\{\beta\}$	= coefficient vector for eigenvector expansion of $\{b\}$
β_i	= elements of $\{\beta\}$
ϵ	= small parameter used in E-D
(Λ)	= diagonal matrix of eigenvalues of (A)
λ_i	= eigenvaue of (A)
$\bar{\lambda}_i$	= eigenvalue of $(\bar{A})$
(Φ)	= matrix of eigenvectors of (A)
$\{\phi_i\}$	= eigenvector of (A)

Introduction

LINEAR algebraic equations with rank deficiency, which are characterized by zero eigenvalues and zero determinants, do not have unique mathematical solutions. Unfortunately, an a priori knowledge of system rank deficiency is not always available, and parameter estimation techniques used in system identification often generate ill-conditioned equations. This can cause anomalous and highly sensitive results, even when special care is taken during the numerical solution process. Such sensitivity is characterized by the fact that small perturbations of the right-hand side or the coefficient matrix (perhaps caused by physical measurement error or limited numerical computation precision) can produce widely varying solutions.

In this Note, matrix eigenvalue theory is used to examine the source of ill conditioning in linear algebraic equations. This approach highlights the crucial role played by the zero and near-zero eigenvalues and corresponding eigenvectors of poorly conditioned systems. Insight gained from this approach is used to significantly improve a recently developed solution procedure called epsilon decomposition (E-D).[1,2] The efficiency of the improved E-D over singular value decomposition (SVD)[3,4] resides in the need to only obtain the zero and near-zero eigenvalues of the coefficient matrix as opposed to all its eigenvalues and vectors (as required by SVD). Thus the efficiency of E-D is significant for large matrices with small rank deficiency.

Technical Background

Given a symmetric set of linear algebraic equations

$$(A)\{x\} = \{b\} \tag{1}$$

where

$$(A) = (A)^T \tag{2}$$

Although an eigenvector solution approach for Eq. (1) is not advocated, it will be useful for future discussions to consider the eigenstructure of (A), $\{x\}$, and $\{b\}$. Therefore, let $\{\phi_i\}$ $i = 1, 2,\ldots, n$ be the orthonormal solutions to

$$(A)\{\phi_i\} = \lambda_i\{\phi_i\} \tag{3}$$

and since (A) is symmetric

$$\begin{aligned} \{\phi_i\}^T\{\phi_j\} &= 0 \qquad i \neq j \\ &= 1 \qquad i = j \end{aligned} \tag{4}$$

Defining the totality of eigenvectors and eigenvalues of Eq. (3) by (Φ) and (Λ), respectively, Eqs. (3) and (4) may be rewritten as

$$(A)(\Phi) = (\Phi)(\Lambda) \tag{5}$$

$$(\Phi)^T = \Phi^{-1} \tag{6}$$

where (Λ) is the diagonal matrix of eigenvalues and the columns of (Φ) are the eigenvectors $\{\phi_i\}$.

Alternate pre- and postmultiplication of Eq. (5) by $(\Phi)^T$ and (Φ), respectively, results in the relationships

$$(\Phi)^T(A)(\Phi) = (\Lambda) \tag{7}$$

$$(A) = (\Phi)(\Lambda)(\Phi)^T \tag{8}$$

whereas eigenexpansion of $\{x\}$ and $\{b\}$ yields

$$\{x\} = (\Phi)\{c\} = \Sigma c_i\phi_i \tag{9}$$

$$\{b\} = (\Phi)\{\beta\} = \Sigma\beta_i\phi_i \tag{10}$$

where $c_i = \{\phi_i\}^T\{x\}$ and $\beta_i = \{\phi_i\}^T\{b\}$.

Substitution of Eqs. (9) and (10) into Eq. (1) and use of Eqs. (6-8) yields

$$(\Lambda)\{c\} = \{\beta\}$$

or

$$\lambda_i c_i = \beta_i \tag{11}$$

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Ill-Conditioning

An ill-conditioned system is popularly defined as one for which solution confidence for Eq. (1) is poor. Examination of Eq. (11) will show that if a direct (e.g., triangular decomposition) solution of Eq. (1) is attempted, when one or more of the λ_i are zero, the results are suspect since the determinant of (A) is zero. This follows from the fact that the determinant of a matrix equals the product of its eigenvalues.

A more formal definition of ill conditioning is when a very large spectral condition number[5] occurs, $K(A)$, where

$$K(A) \equiv \frac{|\lambda_i|\max}{|\lambda_i|\min} \tag{12}$$

If (A) is symmetric, Eq. (11) shows how a very small λ_i and nonzero β_i could produce a signifcant c_i. If very small λ and corresponding β occur due to measurement or computer precision errors, then the corresponding numerical solution $\{\bar{x}\}$ could contain large errors of the form

$$\{\bar{x}\} - \{x\} = \sum_{s=1}^{p} \frac{\beta_s}{\lambda_s}\{\phi_s\} \tag{13}$$

where p is the number of very small nonaccurate eigenvalues, λ_s, and corresponding nonzero right-hand sides β_s.

Singular Value Decomposition

One approach for avoiding the errors associated with ill-conditioned equations is called singular value decomposition (SVD).[3,4] In this approach, the small eigenvalue contributions are discarded from Eq. (9), i.e.,

$$\{x_{\text{SVD}}\} = \Sigma c_i \phi_i \qquad \text{where } i \le n - p$$

or

$$\{x_{\text{SVD}}\} = (\Phi)(\Lambda^*)^{-1}(\Phi)^T\{b\} = \Sigma c_i \phi_i \tag{14}$$

Since

$$(\Lambda^*)^{-1} = \begin{bmatrix} \lambda_1^{-1} & & & 0 & & \\ & \lambda_2^{-1} & & & & 0 \\ & & \ddots & & & \\ 0 & & & & & \\ & & & & \lambda_r^{-1} & \\ \hline & 0 & & & 0 & \end{bmatrix} \tag{15}$$

where r is the rank of matrix (A) and $r = n - p$.

It should be especially noted that the SVD procedure requires the analyst to obtain all nonzero eigenvalues, λ_i, and eigenvectors, $\{\phi_i\}$, of the matrix (A), and that this can become quite costly if (A) is large. As will be shown now, this is not necessary if we employ the method of epsilon decomposition.[2]

Epsilon Decomposition and Its Improvements

Epsilon decomposition (E-D), which is similar to the Levenberg-Marquardt method[6] for stabilizing iterations in nonlinear problems, was first proposed by Ojalvo and Ting[1] for the solution of ill-conditioned linear equations. Unlike SVD, E-D does not require the solution of all the matrix eigenvalues and eigenvectors. Rather, it involves the use of ordinary (e.g., triangular) decomposition to solve the equations:

$$(\bar{A})\{\bar{x}\} = \{b\} \tag{16}$$

where

$$(\bar{A}) = (A) + \epsilon(I) \tag{17}$$

and ϵ is a small scalar.

To understand why $\{\bar{x}\}$ is an approximation to $\{x\}$, we note that the eigenvalues $\bar{\lambda}_i$ and eigenvectors $(\bar{\Phi})$ of $(\bar{A})$ are given by (see Appendix)

$$\bar{\lambda}_i = \lambda_i + \epsilon \tag{18}$$

$$(\bar{\Phi}) = (\Phi) \tag{19}$$

Therefore, a treatment similar to that performed earlier [see Eqs. (3–11)] yields

$$\{\bar{x}\} = \Sigma \bar{c}_i\{\phi_i\} \tag{20}$$

where

$$(\bar{\lambda}_i + \epsilon)\bar{c}_i = \beta_i \tag{21}$$

Thus, if $\epsilon \ll \lambda_i$, then

$$\bar{c}_i \approx c_i$$

In addition, if any λ_s is zero or near zero and the corresponding $\beta_s = 0$, the corresponding $c_s = 0$.

However, if $\beta_s \neq 0$, and $\beta_s \sim 0(\lambda_s)$, then $c_s\{\phi_s\}$ may contribute adversely to $\{\bar{x}\}$, causing it to have a substantial error (departure) from $\{x\}$. A significant improvement to the original Ojalvo-Ting[1,2] E-D solution is possible if the eigenvectors associated with the very small eigenvalues are deleted from $\{\bar{x}\}$ as follows:

$$\{x_{\text{E-D}}\} = \{\bar{x}\} - \Sigma \bar{c}_s \phi_s = \sum_{i=1}^{r} \bar{c}_i \phi_i \tag{22}$$

where

$$\bar{c}_s = \{\phi_i\}^T\{\bar{x}\}, \qquad s = r+1, \ldots, n$$
$$\bar{c}_i = \frac{\beta_i}{\lambda_i + \epsilon}, \qquad i = 1, 2, \ldots, r \tag{23}$$

In this way the errors contributed to $\{\bar{x}\}$, i.e., $\bar{c}_s\{\phi_s\}$, can be filtered out after the $\{\phi_s\}$ are determined.Whereas this improvement requires additional work, it is not significant if the rank deficiency of (A) is small. Furthermore, the efficiency of this improved E-D over SVD is impressive for systems of large rank with no real sacrifice in solution accuracy.

Should it turn out that ϵ is not very small compared to a given eigenvalue, say λ_p, and the corresponding β_p is not zero, then the effect of this term can be included in Eq. (22) by replacing $\bar{c}_p\{\phi_p\}$ by

$$(\bar{c}_p - c_p)\{\phi_p\} = \beta_p\left[\frac{1}{\lambda_p + \epsilon} - \frac{1}{\lambda_p}\right]\{\phi_p\} \tag{24}$$

Thus, the $(\lambda_p + \epsilon)^{-1}$ term removes the erroneous $\{\phi_p\}$ contribution for $\{x\}$ whereas the λ_p^{-1} term properly replaces it in $\{x_{\text{E-D}}\}$.

Numerical Example

Let the matrix (A) be the 4×4 singular matrix

$$(A) = \begin{bmatrix} 1.0596 & 2.1727 & 2.4828 & 4.2386 \\ & 6.5001 & 5.5023 & 8.6910 \\ & & 9.4852 & 9.9307 \\ \text{SYMMETRIC} & & & 16.9547 \end{bmatrix}$$

where the last row equals the first row multiplied by four, and assume we are seeking a solution for $(A)\{x\} = b$ where

$$b = \begin{Bmatrix} 14.1924 \\ 31.5561 \\ 37.3317 \\ 56.7697 \end{Bmatrix} \tag{25}$$

The lowest two eigenvalues of the matrix (A) are $\lambda_0 = 0$ and $\lambda_1 = 1.6046$. The eigenvector corresponding to λ_0 is $\{\phi_0\}$, where

$$\{\phi_0\} = \begin{Bmatrix} -0.9701 \\ 0 \\ 0 \\ 0.2425 \end{Bmatrix}$$

Therefore,

$$\beta_0 = \{\phi_0\}^T\{b\} = 0.00000$$

Thus, we must pick a value of ϵ such that

$$\beta_0 \ll \epsilon \ll \lambda_i \tag{26}$$

or

$$0.00000 \ll \epsilon \ll 1.6046$$

Choosing $\epsilon = 0.00016046$, $(\bar{A})$ becomes

$$(\bar{A}) = (A) + \begin{bmatrix} 0.0002 & 0 & 0 & 0 \\ & 0.0002 & 0 & 0 \\ & & 0.0002 & 0 \\ \text{SYMMETRIC} & & & 0.0002 \end{bmatrix}$$

Solving $(\bar{A})\{\bar{x}\} = \{b\}$ yields

$$\bar{x} = \begin{Bmatrix} 0.6657 \\ 1.0000 \\ 1.0000 \\ 2.0835 \end{Bmatrix} \tag{27}$$

The SVD solution for $\{x\}$ requires that all eigenvalues and eigenvectors be obtained. This yields

$$(\Lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.6046 & 0 & 0 \\ 0 & 0 & 2.7799 & 0 \\ 0 & 0 & 0 & 29.6152 \end{bmatrix}$$

$$(\Phi) = \begin{bmatrix} -0.9701 & -0.1218 & 0.0997 & 0.1845 \\ 0 & 0.8633 & -0.2888 & 0.4139 \\ 0 & 0.0496 & 0.8646 & 0.5000 \\ 0.2425 & -0.4872 & -0.3989 & 0.7380 \end{bmatrix}$$

and

$$\{x_{SVD}\}^T = \lfloor 0.5294, \quad 1.0000, \quad 1.0000, \quad 2.1176 \rfloor \tag{28}$$

However, if we had chosen the right-hand side $\{b'\}$, where

$$\{b'\} = \{b\} + \begin{Bmatrix} -0.0186 \\ 0 \\ 0 \\ 0.0047 \end{Bmatrix}$$

then a solution of $(\bar{A})\{\bar{x}'\} = \{b'\}$ would yield

$$\{\bar{x}'\}^T = \lfloor 9.0996, \quad 1.0001, \quad 0.9996, \quad -0.0248 \rfloor \tag{29}$$

Thus, a small change in $\{b\}$ can produce a dramatic error in $\{\bar{x}'\}$ since $\beta_0 = (\{\phi_0\}^T\{b'\} = 0.0178)$ is not very small compared to ϵ (where $\epsilon = 0.00016$) and we have violated the condition that $\epsilon \gg \beta_0$.

To correct the $\{\bar{x}'\}$ of Eq. (29), we take the next step of E-D, i.e., Eq. (22), to obtain

$$\{x_{E\text{-}D}\} = \{\bar{x}'\} - (-8.8335)\{\phi_0\}$$

$$= \lfloor 0.5302, \quad 0.9999, \quad 1.0004, \quad 2.1173 \rfloor^T \tag{30}$$

which is within a small fraction of 1% of the SVD solution.

A similar correction to Eq. (27) yields

$$\{x_{E\text{-}D}\} = \begin{Bmatrix} 0.6657 \\ 1.0000 \\ 1.0000 \\ 2.0835 \end{Bmatrix} - (\{\phi_0\}^T\{\bar{x}'\})\{\phi_0\} = \begin{Bmatrix} 0.5294 \\ 1.0000 \\ 1.0000 \\ 2.1176 \end{Bmatrix} \tag{31}$$

which is the identical solution as that obtained by SVD.

A summary of these numerical results is presented in Table 1.

Thus, the refinement proposed by Eq. (22) changes the original E-D solution given by $\{\bar{x}'\}$ dramatically and yields a value that is very close to the SVD solution without requiring the computation of all the eigenvalues and eigenvectors of (A) when it is singular.

Appendix

Given a system defined by Eqs. (3–6) and a second system defined by Eq. (17), then

$$(\bar{A}) = (A) + \epsilon(I) = (\Phi)(\Lambda)(\Phi)^T + \epsilon(\Phi)(\Phi)^T \tag{A1}$$

but

$$(\bar{A}) = (\bar{\Phi})(\bar{\Lambda})(\bar{\Phi})^T \tag{A2}$$

Therefore,

$$(\bar{\Phi})(\bar{\Lambda})(\bar{\Phi})^T = (\Phi)[(\Lambda) + \epsilon(I)](\Phi)^T \tag{A3}$$

$$(\bar{\Phi}) = (\Phi) \tag{A4}$$

Table 1 Solutions to numerical example

Variable	SVD [see Eqs. (14) and (28)]	E-D without Eq. (22) improvement [see Eqs. (16) and (27)]	E-D with Eq. (22) improvement [see Eq. (31)]	E-D without Eq. (22) improvement and with RHS[a] contamination [see Eq. (29)]	E-D with Eq. (22) improvement and RHS[a] contamination [see Eq. (30)]
1	0.5294	0.6657	0.5294	9.0996	0.5302
2	1.0000	1.0000	1.0000	1.0001	0.9999
3	1.0000	1.0000	1.0000	0.9996	1.0004
4	2.1176	2.0835	2.1176	−0.0248	2.1173

[a]RHS = right-hand side.

and

$$(\bar{\Lambda}) = (\Lambda) + \epsilon(I) \tag{A5}$$

or

$$\bar{\lambda}_i = \lambda_i + \epsilon \tag{A6}$$

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